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Abstract: We consider an economy characterized by the simultaneous presence of public goods and many increasing returns to scale firms. We formalize a notion of equilibrium, termed a Lindahl-Hotelling equilibrium, in which producer prices are marginal cost prices and the public goods are financed through Lindahl prices. Sufficient conditions are provided for such an equilibrium to exist.

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1. Introduction

In this paper we consider an equilibrium concept for an economy characterized by the simultaneous presence of public goods and many increasing returns to scale firms. In the equilibrium notion that we study, all firms facing increasing returns to scale are regulated to follow marginal cost pricing, with their losses financed through given rules for lump sum taxation, consumers maximize utility, firms facing decreasing or constant returns to scale maximize profits at the market prices and the marginal cost of each public good is defrayed among the consumers through Lindahl prices. We present sufficient conditions under which such an equilibrium, to be termed a Lindahl - Hotelling equilibrium, exists. Our equilibrium concept originates in Lindahl (1919) and Hotelling (1938); is alluded to by Bowen (1943); and is implicit in a somewhat neglected contribution of Henderson (1947).

A generalized version of the second welfare theorem pertaining to economies with public goods and increasing returns is presented in Khan - Vohra (1987) where it is shown that *if* arbitrary lump sum taxes are feasible, any Pareto optimal allocation of an economy with public goods and increasing returns can be sustained by marginal cost prices and Lindahl prices, i.e. as a Lindahl - Hotelling equilibrium. In this paper we show the existence of a Lindahl - Hotelling equilibrium when arbitrary lump sum transfers are not allowed and the rules for income distribution are fixed. It should, however, be emphasized that in an economy with increasing returns and fixed rules for income distribution, even without public goods, and despite a general version of the second welfare theorem, marginal cost pricing equilibria may not have very attractive welfare properties, as is borne out by the examples in Guesnerie (1975), Brown - Heal (1979), Beato - Mas-Colell (1985) and Vohra (1985). In particular, none of these equilibria need be Pareto optimal.

Next, we turn to a discussion of the antecedent literature. In the seminal work of Hotelling (1938,1939) one can discern two distinct ideas. The first represents the pricing of public utilities which are taken as examples of firms with increasing returns to scale in production. Hotelling argued that marginal cost pricing was necessary for Pareto optimality. This aspect of Hotelling's work has now been formalized in the form of a general theorem on the existence of marginal cost pricing equilibria and a generalized second welfare theorem for economies with arbitrary production sets. For the former see Beato - Mas-Colell (1985), Brown *et al* (1986), Bonnisseau - Cornet (1986) and Vohra (1986), while for the latter, see Guesnerie (1975), Khan - Vohra (1987), Cornet (1986)

and the references therein.

The second important idea in Hotelling's work relates to the financing of what we can recognize, after Samuelson (1954), as public goods. Hotelling illustrated the heuristics of his argument in terms of Dupuit's celebrated example of the bridge, which can be seen as a public good. Equivalently, a bridge has the property that the marginal cost of letting an additional person use it is zero. This definition entailed as a direct consequence, that the marginal cost pricing principle applied as much to economies of scale associated with the consumption of public goods as it did to economies of scale in production. The distinction between these two different concepts was not emphasized, and deservedly so, in a work whose primary emphasis was on showing that tolls or excise taxes, or equivalently average cost pricing, were detrimental to general welfare. The distinction becomes important only when price discrimination is feasible as a means for financing the difference between average and marginal costs or when the quantity of the public good is to be concurrently determined. But, of course, on this Hotelling was explicitly silent.

It is only in the later work of Bowen (1943) and Henderson (1947) that one comes close to a discussion of a simultaneous resolution of both problems. Bowen (1943) pointed out that if the public good was produced under conditions of decreasing costs, optimality would require that the Lindahl prices added up to the marginal cost, leading to losses which would have to be financed in some other way. As alternatives for covering the losses of the increasing returns industry, Henderson (1947) considers both lump-sum taxation *a la* Hotelling and two-part tariffs. In the context of public goods this turns out to be similar to the solution formulated by Lindahl (1919) in the simpler setting of an economy without increasing returns to scale; since *all* consumers consume a public good, Lindahl taxes can be seen simply as the fixed part of Henderson's two-part tariff with the variable part being zero. Notice, however, that the fixed part may vary from consumer to consumer. Moreover, this similarity disappears in an economy with increasing returns where all losses (even those of firms producing public goods) are collected not through endogenously determined fixed parts but rather through exogenously given rules for lump sum taxation - as is the case in the equilibrium notion that we consider.

One can conceive of other interesting notions of equilibrium in an economy with increasing returns and public goods, especially given the fact that a Lindahl - Hotelling equilibrium may not be Pareto optimal. In Vohra (1983) an equilibrium concept was considered where both, the losses

of the increasing returns sector and the cost of public goods, were financed through lump-sum taxes based on the wealth of the consumers. Such a solution concept is purely Hotelling's; marginal cost pricing is applied to production as well as to the provision of public goods (the marginal cost in the latter case being zero). This form of taxation may, however, change the initial distribution of income. Indeed, this was one of the objections Frisch (1939) raised against Hotelling's notion of equilibrium. In a Lindahl - Hotelling equilibrium some of the arbitrariness of taxation is attenuated simply because Lindahl prices are based on the benefit principle. Pushing this possibility further, an intriguing possibility arises in the context of economies where the public good is being produced under increasing returns to scale; namely, whether this arbitrariness can be can be completely eliminated by assigning the losses in the same proportion as the personalized prices. Such a notion of equilibrium has been precisely formulated and argued for by Kaneko (1977) in the context of economies with convex production sets. For economies with increasing returns this approach has been followed by Mas-Colell - Silvestre (1985).

We now turn to the technical contribution of this paper. Foley (1970) showed that the existence of Lindahl equilibrium could be shown using standard arguments if the dimensionality of the underlying space was suitably expanded. In particular, he augmented the aggregate production set by assigning the public goods output to as many coordinates as there were consumers. The augmented production set therefore belongs to an $n + l$ dimensional subspace of R^{n+Tl}, where n, l and T refer to the number of private goods, public goods and consumers respectively. On the other hand, proofs of existence of marginal cost pricing equilibria in non-convex economies rely crucially on a homeomorphism between the production frontier and the simplex corresponding to its underlying space. Clearly, a direct application of Foley's approach would have to be rejected in this context since the boundary of the augmented production set, belonging to an $(n + l - 1)$ dimensional subspace of R^{n+Tl}, cannot be homeomorphic to the $(n + Tl - 1)$ dimensional simplex. We therefore adopt an approach based on earlier work (Khan - Vohra (1985)) pertaining to convex economies, which does not require augmenting the production set.

In section 2 we present the formal model and result. Section 3 provides a heuristic overview of the existence proof and the technically inclined reader may skip this section and go, without any loss of continuity, directly to section 4 which contains the proof. We end the paper with some concluding remarks in section 5.

2. The Model and Result

We consider an economy with T consumers, each consumer t having a consumption set $X^t \subseteq R_+^{n+l}$. The projections of X^t onto the space of private and public goods are denoted $X_\pi^t \subseteq R^n$ and $X_g^t \subseteq R^l$ respectively. Clearly X_g^t must be identical for all t and we can therefore write it as X_g. Let $x^t = (x_\pi^t, x_g^t)$ refer to a point in $X^t = X_\pi^t \times X_g^t$. The preferences of consumer t are represented by a continuous utility function $U^t(.) : X^t \mapsto R$. The endowment of consumer t is denoted $\omega^t \in X_\pi^t$. The aggregate endowment $\omega \in R^{n-l}$ is defined as $\omega = (\sum_t \omega^t, 0)$. We shall consider H firms which have production sets $Y^h \subseteq R^{n+l}$, $h = 1,.,.,H$, which are not necessarily convex. Since we shall be assuming free disposal, there is no loss of of generality in assuming that there also exists a competitive firm with a convex production set $Y^c \subseteq R^{n+l}$. We shall use j as the index for firms in general; $j = 1,\ldots,H,c$. It will also be convenient to use h as an index for the non convex firms. Thus, $\sum_h$ refers to summation only across firms $1,\ldots,H$, while $\sum_j$ refers to summation across all firms $1,\ldots,H,c$. A consumption plan is $(x^t) \in \prod_t X^t$ and a production plan is $(y^j) \in \prod_j Y^j$. We shall use x_π to denote $\sum_t x_\pi^t$ y to denote $\sum_j y^j$ and Y to denote $\sum_j Y^j$. Throughout, we shall use superscripts on variables to index the agents. Subscript i will be used to index commodities; for any vector $z \in R_+^{n+l}$, z_i denotes the i-th element of z. Sometimes i will be used to index all commodities and sometimes only to index the various public goods, but no confusion should arise. For z, $\bar{z} \in R^{n+l}$ $z >> \bar{z}$ means that $z_i > \bar{z}_i$ for all i and $z > \bar{z}_i$ means that $z_i \geq \bar{z}_i$ for all i and one of the inequalities is strict.

Market prices are denoted $p = (p_\pi, p_g)$ where p_π and p_g refer to the prices of the private and the public goods respectively. The Lindahl price of the i-th public good for consumer t is $p_{g_i}^t = s_i^t p_{g_i}$, where s_i^t is the Lindahl share of the t-th consumer for the i-th public good. The vector of Lindahl shares for the i-th public good is denoted s_i and is an element of the (T-1) dimensional simplex S, i.e. $\sum_t s_i^t = 1$, $s_i^t \geq 0$, for all i, $i = n+1,\ldots,n+1$. The l-fold Cartesian product of S is denoted S^l. We denote by p_g^t the vector of Lindahl pices for consumer t and $p^t = (p_\pi, p_g^t)$ is the vector of prices facing consumer t.

Let $\triangle$ be the unit simplex in R^{n+l}. Given a vector of reference prices $p \in \triangle$ and an efficient production plan $(y^j) \in \prod_j Y^j$, the income of consumer t is defined by a continuous function $r^t(p, (y^j))$, where $\sum_t r^t(p, (y^j)) = p \cdot (y + \omega)$ and the budget correspondence is denoted $\gamma^t(p, (s_i), (y^j)) = \{ x^t \in X^t \mid p^t \cdot x^t \leq r^t(p, (y^j)) \}$. In a private ownership economy *a la* Arrow-Debreu,

where each consumer t has a share θ^{tj} in firm j and endowment ω^t, $r^t(p,(y^j)) = \sum_j \theta^{tj} p\cdot y^j + p\cdot(\omega^t, 0)$. If the income is given by a *fixed structure of revenues*, $r^t(p,(y^j)) = a^t(p\cdot(y+\omega))$, where $a^t > 0$ for all t and $\sum_t a^t = 1$.

To formally develop the notion of marginal costs or marginal rates of transformation, we shall need the following definitions:

Definition 1. The *Tangent Cone* to the set $Y \subseteq R^{n+l}$ at the point $y \in Y$ is defined as

$$T(Y,y) = \{x \in R^{n+l} \mid \text{ for every } y^k \in Y,\ y^k \to y \text{ and every } t^k \in (0,\infty),\ t^k \to 0,$$
$$\text{there exists } x^k \in R^{n+l},\ x^k \to x, \text{ such that } y^k + t^k x^k \in Y \text{ for all } k\}.$$

Definition 2. The *Normal Cone* to the set Y at the point $y \in Y$ is defined as

$$N(Y,y) = \{x \in R^{n+l} \mid x\cdot z \leq 0 \text{ for all } z \in T(Y,y)\}.$$

The set of marginal cost prices to Y at y is the set $N(Y,y)\cap\Delta$. This is the definition of marginal cost prices which, by now is standard in the recent literature (see for example, Khan - Vohra (1987)). If Y is convex, then $N(Y,y)\cap\Delta$ is precisely the set of all normalized prices which maximize profits over Y at y.

We can now formally define a *Lindahl - Hotelling Equilibrium.*

Definition 3. *A Lindahl - Hotelling Equilibrium* is defined as $((\bar{x}^t_\pi), \bar{x}_g, (\bar{y}^j), \bar{p}, (\bar{s}_i)) \in \prod_t X^t_\pi \times X_g \times \prod_j Y^j \times \Delta \times S^l$ such that

(i) For all t, $(\bar{x}^t_\pi, \bar{x}_g) \in \gamma^t(\bar{p}, (\bar{s}_i), (\bar{y}^j))$ and $U^t(\bar{x}^t_\pi, \bar{x}_g) \geq U^t(x^t)$ for all $x^t \in \gamma^t(\bar{p}, (\bar{s}_i), (\bar{y}^j))$,

(ii) For all j, $\bar{p} \in N(Y^j, \bar{y}^j)$,

(iii) $(\bar{x}_\pi, \bar{x}_g) = \bar{y} + \omega$.

In words, a Lindahl - Hotelling equilibrium consists of consumption plans, with each consumer consuming the same amount of public goods; production plans; a price system; and consumers' shares in the cost of each public good; such that

(i) each consumption plan is utility maximizing in a budget set based on a modified price system reflecting personalized (Lindahl) prices for the public goods,

(ii) the price system is a vector of normalized, marginal cost prices for each non convex firm at its production plan,

(iii) each convex firm maximizes profits at its production plan.

(iii) the aggregate demand for each private good is equal to its aggregate net supply and the common demand for each public good is equal to its aggregate net supply.

For our existence result we shall need the following assumptions.

A1. for all t, X^t is a closed, convex subset of R_+^{n+l} and contains 0; $U^t(.)$ are continuous, quasi-concave and satisfy local non-satiation; for all t if $\bar{x}_g, x_g \in X_g$ and $\bar{x}_g > x_g$ then $U^t(x_\pi^t, \bar{x}_g) \geq U^t(x_\pi^t, x_g)$.

A2. Y^c is convex. For all $j = 1, ., ., H, c$, Y^j is closed, contains 0 and satisfies free disposal ,i.e. $Y^j - R_+^{n+l} \subseteq Y^j$.

A3. Y is closed and $\mathcal{A}(Y) \cap -\mathcal{A}(Y) = \{0\}$, where $\mathcal{A}(Y)$ denotes the asymptotic cone of Y.

A4. $r^t(p, (y^j))$ is a continuous function for all t and if $p \in N(Y^j, y^j)$ for all j, then $\sum_t r^t > 0$ implies that $r^t > 0$ for all t.

These assumptions are standard and given these it can be shown the there exists a cube $K = \{z \in R^{n+l} \mid |z_i| \leq k, \text{ for all } i = 1, \ldots, n+l\}$, which contains in its interior all the attainable consumption and production sets (the argument is presented in the proof of the Theorem below). Let $\partial(Y^h)$ denote the boundary of Y^h and e the vector in R^{n+l} all of whose coordinates are 1. We now translate the production set for each h and consider a certain subset of its boundary, which we will later show to be homeomorphic to the simplex Δ. Let

$$E(Y^h) = \left(\partial(Y^h + \{ke\})\right) \bigcap R_+^{n+l}, \quad h = 1, \ldots, H.$$

We shall also make the following assumption.

A5. (i) For all $h = 1, \ldots, H$ and all $i = 1, \ldots n+l$, if $y_i^h = -k$, and $q^h \in N(Y^h, y^h)$, then $q_i^h = 0$.
(ii) If $(y^h) \in (E(Y^h - ke))$, $y^c \in Y^c$ and $p \in N(Y^j, y^j)$ for all j, then $p \cdot (y + \omega) > 0$.

A5(i) states that when the output of commodity i is $-k$, which is beyond the bounds of the attainable set, the marginal cost price of i is 0. Since this is a condition on production plans which are not in the attainable set, it can be imposed without any loss of generality (see Vohra (1986)

for details). The significant part of A5 is A5(ii) which requires that for all production plans (on the boundary of the truncated production sets for non convex firms) at which there there exists a common marginal cost price for all firms, the aggregate income of the economy, evaluated at this price system, is positive. In a private goods economy with convex production sets and $\omega >> 0$, this assumption is automatically satisfied. In the context of economies with increasing returns, this kind of assumption is quite common (see for example, Beato - Mas-Colell (1985) and Brown *et al* (1986)).

We can now state the main result of this paper, the proof of which is provided in section 4.

Theorem. *If (A1)-(A5) are satisfied, then there exists a Lindahl - Hotelling equilibrium.*

3. An Overview of the Existence Proof

As is usual for existence proofs in economics, ours is based on an application of Kakutani's fixed point theorem. Thus, we construct a non empty, convex, compact set and a non empty, upper hemicontinuous, convex valued correspondence from this set to itself whose fixed point yields a Lindahl - Hotelling equilibrium. However, the argument differs enough from the standard treatment, as in Debreu (1959), that a heuristic introduction to the basic ideas seems warranted.

Three basic difficulties have to be faced at the outset. The first of these relates to the compactness of the attainable set in a finite but non convex economy. This is overcome by the observation, originally due to Hurwicz - Reiter (1973), that given irreversibility (in a modified form) and other standard assumptions, Debreu's (1959) argument on the compactness of the attainable set does not hinge on the convexity hypothesis on production sets.

The second difficulty has to do with the non convexity of the production sets themselves. This is overcome by the observation that in the presence of free disposal, the boundary of a suitably truncated production set is, for all purposes of the proof, "identical" to the simplex, i.e. there is a homeomorphism from this truncated boundary to the simplex. The truncations are chosen in light of the compactness of the attainable set, in particular, to guarantee that the fixed point does not involve production plans on the boundary of the simplex.

The above difficulties are, by now, well understood in the literature on the existence of marginal cost pricing equilibrium; for details and references the reader may see, for example, Beato - Mas-Colell (1985), Bonnisseau - Cornet (1986), Brown *et al* (1986) and Vohra (1986).

The third difficulty relates to public goods in an economy with non convex production sets. Since we make use of a homeomorphism between the boundary of a production set and the simplex, as mentioned in section 1, we can no longer rely on Foley's (1970) approach of considering augmented production sets which belong to an $(n+l-1)$ dimensional subspace of R^{n+Tl}. We circumvent this problem by considering an alternative way of obtaining Lindahl shares as described in (d) below.

We shall now describe the construction of the correspondence. Recall that we work with T consumers indexed by t, H non convex firms and one convex firm, generally indexed j, $j=1,\ldots,H,c$ and indexed by h when only the non convex firms are being considered, n private goods and l public goods all indexed by i, or when the public goods are to be emphasized, by g_i. The domain, and the range, of the correspondence is a Cartesian product of the following sets:

(i) $H+1$ simplices $\triangle$, all in R^{n+l}, and each serving as a proxy for the relevant boundary of a particular production set. Note that we also proxy the convex production set.

(ii) $H+1$ simplices $\triangle$, each reflecting the set which contains the normalized marginal cost prices for the corresponding production set.

(iii) T consumption sets $\bar{X}^t$, all subsets of R_+^{n+l}, and all truncated in light of the compactness of the attainable set.

(iv) l simplices S, all in R^T, so that a point in S specifies the Lindahl shares of the consumers for a particular public good.

In summary, the domain and range of our correspondence is given by

$$\Gamma = \triangle^{H+1} \times \triangle^{H+1} \times \prod_t \bar{X}^t \times S^l.$$

We now turn to the correspondence itself which is the product of five basic correspondences:

(a) The first set of $H+1$ correspondences μ^j are marginal cost pricing correspondences which take a point from $\triangle$ to a particular production plan via the homeomorphism and then to the normalized normal cone.

(b) The second set of T correspondences ξ^t are modified demand correspondences and take prices, as given by the marginal costs of the convex firm, Lindahl shares and incomes (computed at these prices and the given production plans) to the set of utility maximizing consumption plans in the truncated consumption sets. This is a standard map in general equilibrium theory, the only difference being that it is not being aggregated across the consumers and

that demand is equated to zero if income is negative - a possibility precluded in Debreu (1959). It is worth mentioning that it is the presence of public goods which prompts us to consider a Cartesian product of ξ^t rather than a sum. Note also that the profits of the firms are being computed at the marginal cost prices of the convex firm.

(c) The third set of of H mappings β^h are functions pertaining to the adjustment of production plans for the non convex firms and each takes the given production plan of the firm, its marginal cost and the marginal cost of the convex firm to a new production plan, i.e. it takes an element in $\triangle \times \triangle \times \triangle$ to $\triangle$. As in the correspondences under (b) these functions also view the marginal cost price of the convex firm as the basic reference price system. The production of a commodity is increased, in relative terms, if the reference price of the commodity exceeds the marginal cost of the commodity for the non convex firm. This serves to select production plans at which all firms have a common vector of marginal cost prices. For the precise, and intuitively appealing, formula see the definition of β^h in the next section.

(d) Our next set of l functions ψ_i pertain to the adjustment of consumers' Lindahl shares in the cost of provision of public goods. Each takes the quantity of the i-th public good demanded by the various consumers and their given Lindahl shares for this commodity into new Lindahl shares. In particular, a consumer's share in cost of a public good is increased, in relative terms, if his demand for that good exceeds a weighted average demand. This weighted average is computed as a weighted sum of the demands for that good, the weights being the corresponding, given Lindahl shares. These functions do for the Lindahl shares what the mappings under (c) do for the production plans of the non convex firms; they serve to adjust these shares so as to make the demands, of the various consumers, for a public good identical. For the formula, see the next section.

So far we have adjusted the marginal cost prices of each firm (under (a)), consumer demands (under (b)), the production plans of non convex firms (under (c)) and Lindahl shares (under (d)). The only element which remains unadjusted is the production plan of the convex firm. In more technical terms, the product of our mappings takes Γ into a set which differs from Γ by $\triangle$. This is remedied next.

(e) Our final mapping ϕ can be seen as a function which adjusts the production plan of the convex firm in accordance with the aggregate excess demand from the rest of the economy.

The specification of the correspondences is now complete. One only has to check that each of these satisfy the conditions of Kakutani's fixed point theorem and that one can obtain from a fixed point a Lindahl - Hotelling equilibrium. We defer this to the formal proof below and conclude this section with the following remarks.

First, it is to be emphasized that we are describing an existence proof and whereas words such as "adjustment" aid intuitive understanding, there is no presumption as to any dynamic process converging to an equilibrium. Second, the mappings under (c), (d) and (e) all fall under the general class of "penalty maps" and go back to Nash (1952, p. 288). In our specific context, those under (c) are available in Beato - Mas-Colell (1985), those under (d) in Khan - Vohra (1985) and that under (e) in Brown *et al* (1986).

4. Proof of the Theorem

In order to simply the proof, we shall actually show that there exists $((\bar{x}^t_\pi), \bar{x}_g, (\bar{y}^j), \bar{p}, (\bar{s}_i)) \in \prod_t X^t_\pi \times X_g \times \prod_j Y^j \times \triangle \times S^l$ satisfying conditions (i) and (ii) of Definition 3 and

(iii)' $(\bar{x}_\pi, \bar{x}_g) \leq \bar{y} + \omega$.

Given local non satiation, free disposal and the convexity of Y^c we can then follow the usual argument, as in Debreu (1959, p. 86,87), to construct another allocation which satisfies conditions (i), (ii) and (iii). Let $\hat{y}^c = \bar{y}^c + ((\bar{x}_\pi, \bar{x}_g) - (\bar{y} + \omega))$ which, by free disposal, belongs to Y^c. Since utility functions satisfy local non satiation, condition (i) implies that $\bar{p} \cdot (\bar{x}^t_\pi, \bar{x}_g) = \bar{p} \cdot (\bar{y} + \omega)$, i.e. $\bar{p} \cdot \bar{y} = \bar{p} \cdot \hat{y}$. Since Y^c is convex and, by condition (ii), $\bar{p} \in N(Y^c, \bar{y}^c)$, this also implies that $\bar{p} \in N(Y^c, \hat{y}^c)$. It is now straightforward to check that $((\bar{x}^t_\pi), \bar{x}_g, (\bar{y}^h), \hat{y}^c, \bar{p}, (\bar{s}_i))$ is a Lindahl - Hotelling Equilibrium. Given this observation, we shall now confine ourselves to the case in which condition (iii) of Definition 3 holds with a weak inequality.

We begin by verifying that the attainable set A is compact, where

$$A = \{ ((x^t), (y^j)) \in \prod_t X^t \times \prod_j Y^j \mid x^t_g = x_g \text{ for all } t \text{ and } (x_\pi, x_g) = y + \omega \}.$$

Consider the set

$$\bar{A} = \{ ((x^t_\pi), x_g, (y^j) \in \prod_t X^t_\pi \times X_g \times \prod_j Y^j \mid (x_\pi, x_g) = y + \omega \}.$$

We can appeal to lemma 1 in Brown *et al* (1986) to assert that $\bar{A}$ is compact. But A can be written as

$$A = \{((x_\pi^t, x_g), (y^j) \in \prod_t X^t \times \prod_j Y^j \mid ((x_\pi^t), x_g, (y^j)) \in \bar{A}\}.$$

A can, therefore, be obtained from $\bar{A}$ simply by introducing extra coordinates and since $\bar{A}$ is compact so is A. Thus, there exists $k > 0$ such that K, the cube with edge $2k$, contains in its interior all the attainable sets. Given this k, let

$$f = (H + T + 1)ke + \omega,$$

$$\bar{X}^t = X^t \cap K, \qquad \text{for } 1 = 1, \ldots, T,$$

$$E(Y^c) = \big(\partial(Y^c + f)\big) \bigcap R_+^{n+l},$$

$$E(Y^h) = \big(\partial(Y^h + \{ke\})\big) \bigcap R_+^{n+l}, \qquad \text{for } h = 1, \ldots, H.$$

For $j = 1, \ldots, H, c$, let

$$\bar{E}(Y^j) = \{z \in E(Y^j) \mid \nexists z' \in E(Y^j) \text{ such that } z' \leq z \text{ and } z'_i < z_i \text{ for all } i \text{ for which } z_i > 0\}.$$

We can appeal to lemma 2 of Brown *et al* (1986), to assert that $\bar{E}(Y^j)$ is homeomorphic to $\triangle$ for all j. Moreover, the homeomorphism can be defined by a function $\nu^j : \triangle \mapsto \bar{E}(Y^j)$ such that $\nu^j(\delta) = t\delta$ for some $t > 0$. Thus $\nu^j(\delta)_i > 0$ if and only if $\delta_i > 0$. Let

$$y^c(\delta^c) = \nu^c(\delta^c) - f \qquad \text{and} \qquad y^h(\delta^h) = \nu^h(\delta^h) - ke \quad \text{for} \quad h = 1, \ldots, H.$$

We now define the mapping which associates marginal cost prices with a production plan of firm $j, j = 1, \ldots, H, c$, as $\mu^j : \triangle \mapsto \triangle$, where $\mu^j(\delta^j) = N(Y^j, y^j(\delta^j)) \cap \triangle$. It can be shown that μ^j is non-empty, convex-valued and upper-hemicontinuous (see the proof of Theorem 1 in Brown *et al* (1986)).

For $(p, s, \delta) \in \triangle \times S^l \times \triangle^{H+1}$, let $\bar{\gamma}^t(p, s, \delta) = \gamma^t(p, (s_i), (y^j(\delta^j)) \cap \bar{X}^t$. We can now define the modified demand correspondence for consumer t as $\xi^t : \triangle \times S^l \times \triangle^{H+1} \mapsto \bar{X}^t$, where

$$\xi^t(p, s, \delta) = \begin{cases} \{x^t \in \bar{\gamma}^t(p, s, \delta) \mid U^t(x^t) \geq U^t(x'^t) \text{ for all } x'^t \in \bar{\gamma}^t(p, s, \delta)\} & \text{if } r^t(p, (y^j(\delta^j))) > 0; \\ \bar{\gamma}^t(p, s, \delta) & \text{if } r^t(p, (y^j(\delta^j))) = 0; \\ 0 & \text{otherwise.} \end{cases}$$

By A1., this correspondence is non empty, convex valued and upper hemicontinuous for all t.

To ensure that all producer prices are identical to the reference prices we define for each non convex firm h , a mapping $\beta^h : \Delta^3 \mapsto \Delta$ which adjusts its production according to the deviation between its price q^h and the reference price p. For $h = 1, \ldots, H$, β^h is defined as follows:

$$\beta_i^h(q^h, p, \delta^h) = \frac{\delta_i^h + \mathrm{Max}(0, p_i - q_i^h)}{\sum_{m=1}^{n+l}\left(\delta_m^h + \mathrm{Max}(0, p_m - q_m^h)\right)}, \qquad i = 1, \ldots, n + l.$$

Certainly, β^h is a continuous function for all $h = 1, \ldots, H$.

To ensure that all consumers demand identical amounts of public goods we define a penalty mapping which adjusts the Lindahl shares depending on the deviation between a consumer's demand and the corresponding weighted average. For public good i, $i = n+1, \ldots n+l$, let

$$\tilde{x}_{g_i} = \sum_t s_i^t x_{g_i}^t$$

and define $\psi_i : S_i \times \prod_t \bar{X}_g^t \mapsto S_i$, where

$$\psi_i^t(s_i, (x_g^t)) = \frac{s_i^t + \mathrm{Max}(0, x_{g_i}^t - \tilde{x}_{g_i})}{\sum_{m=1}^{T}\left(s_i^m + \mathrm{Max}(0, x_{g_i}^m - \tilde{x}_{g_i})\right)}, \qquad t = 1, \ldots, T.$$

Clearly, for all i, $i = n+1, \ldots n+l$, ψ_i is a continuous function.

The production of firm c is adjusted in accordance with the aggregate excess demand of the rest of the economy. Let $\phi : \prod_t \bar{X}_\pi^t \times \bar{X}_g \times \Delta^H \mapsto \Delta$ be defined such that

$$\phi((x_\pi^t), x_g, (\delta^h)) = (1/\lambda)\Big((x_\pi, x_g) - \sum_h y^h(\delta^h) - \omega + f\Big),$$

where

$$\lambda = \sum_{i=1}^{n+l}\Big((x_\pi, x_g) - \sum_h y^h(\delta^h) - \omega + f\Big)_i.$$

From the definition of k and f it is clear that $\big((x_\pi, x_g) - \sum_h y^h(\delta^h) - \omega\big) >> -f$. Thus ϕ is a well defined, continuous function.

Finally, let $\alpha : \Delta^{H+1} \times \Delta^{H+1} \times \prod_t \bar{X}^t \times S^l \mapsto \Delta^{H+1} \times \Delta^{H+1} \times \prod_t \bar{X}^t \times S^l$ be defined by

$$\alpha((q^h), p, (\delta^j), (x^t), (s_i)) = \prod_j \mu^j(\delta^j) \times \prod_h \beta^h(q^h, p, \delta^h) \times \phi \times \prod_t \xi^t(p, s, \delta) \times \prod_{i=n+1}^{n+l} \psi_i(s_i, (x_g^t)).$$

Clearly α satisfies the conditions of Kakutani's fixed point theorem and therefore has a fixed point $((\bar{q}^h), \bar{p}, (\bar{\delta}^j), (\bar{x}^t), (\bar{s}_i))$, where

$$\bar{q}^h \in \mu^h(\bar{\delta}^h) \qquad h = 1, \ldots, H, \qquad \bar{p} \in \mu^c(\bar{\delta}^c),$$

$$\bar{\delta}^h \in \beta^h(\bar{q}^h, \bar{p}, \bar{\delta}^h), \qquad h = 1, \ldots, H, \qquad \bar{\delta}^c \in \phi((\bar{x}^t_\pi), \bar{x}_g, (\bar{\delta}^j))$$

$$\bar{x}^t \in \xi^t(\bar{p}, \bar{s}, \bar{\delta}) \qquad t = 1 \ldots, T,$$

and

$$\bar{s}_i \in \psi_i(\bar{s}_i, (\bar{x}^t_g)) \qquad i = n+1, \ldots, n+l.$$

We shall now show that corresponding to this there exists $((\bar{x}^t_\pi), \bar{x}_g, (\bar{y}^j), \bar{p}, (\bar{s}_i))$, which satisfies conditions (i) and (ii) of Definition 3 and condition (iii)', where $\bar{x}_{g_i} = \sum_t \bar{s}^t_i \bar{x}^t_{g_i}$ and $\bar{y}^j = y^j(\bar{\delta}^j)$.

From the mapping β^h it follows that

(1) $$\bar{\delta}^h_i \sum_{m=1}^{n+l} \left(\text{Max}(0, \bar{p}_m - \bar{q}^h_m)\right) = \text{Max}(0, \bar{p}_i - \bar{q}^h_i), \qquad i = 1, \ldots, n+l.$$

This yields the following condition:

(2) If $\bar{p} \neq \bar{q}^h$, then $\bar{p}_i > \bar{q}^h_i$ for i such that $\bar{\delta}^h_i > 0$, and $\bar{p}_i \leq \bar{q}^h_i$ for i such that $\bar{\delta}^h_i = 0$.

Given A5 (i), we know that if $\bar{\delta}^h_i = 0$ then $\bar{q}^h_i = 0$. Given that $\bar{p}$, $\bar{q}^h_i \in \triangle$ it now follows from (2) that $\bar{q}^h = \bar{p}$. The definition of μ^j now implies that $\bar{p} \in N(Y^j, \bar{y}^j)$ for all j. Thus condition (ii) of equilibrium is satisfied. Notice that, from A4 and A5 (ii), this also implies that $r^t(\bar{p}, (\bar{y}^j)) > 0$ for all t.

Since we know that $r^t(\bar{p}, (\bar{y}^j)) > 0$ for all t, it follows from the definition of ξ^t that $(\bar{x}^t_\pi, \bar{x}^t_g) \in \bar{\gamma}^t(\bar{p}, \bar{s}, \bar{\delta})$ and maximizes utility over this truncated budget set. We shall now use the properties of the mappings ξ^t and ψ^t to show that condition (i) of equilibrium is satisfied by $(\bar{x}^t_\pi, \bar{x}_g)$ for all t. From ψ_i and the definition of $\bar{x}_{g_i}$ it follows that for all t,

(3) $$\bar{s}^t_i \sum_{m=1}^{T} \left(\text{Max}(0, \bar{x}^m_{g_i} - \bar{x}_{g_i})\right) = \text{Max}(0, \bar{x}^t_{g_i} - \bar{x}_{g_i}), \qquad t = 1, \ldots, T.$$

Suppose that for some τ, $\bar{x}^\tau_{g_i} \neq \bar{x}_{g_i}$. If $\bar{s}^\tau_i > 0$, then, given the definition of $\bar{x}_{g_i}$, it must be the case that $\sum_{m=1}^{T} \left(\text{Max}(0, \bar{x}^m_{g_i} - \bar{x}_{g_i})\right) > 0$. Now (3) implies that $\bar{x}^t_{g_i} > \bar{x}_{g_i}$ for all t such that $\bar{s}^t_i > 0$. But this contradicts the definition of $\bar{x}_{g_i}$. Thus,

(4) $$\bar{x}^t_{g_i} = \bar{x}_{g_i} \text{ for all } t \text{ such that } \bar{s}^t_i > 0.$$

And, it follows directly from (3) that

(5) $$\bar{x}^t_{g_i} \leq \bar{x}_{g_i} \text{ for all } t \text{ such that } \bar{s}^t_i = 0.$$

Thus, for no consumer is $\bar{x}^t_{g_i}$ greater than $\bar{x}_{g_i}$ and it is only for those consumers who face a zero Lindahl price for i that it can be less. Since $(\bar{x}^t_\pi, \bar{x}^t_g) \in \bar{\gamma}(\bar{p}, \bar{s}, \bar{\delta})$ for all t, this implies that $(\bar{x}^t_\pi, \bar{x}_g) \in \bar{\gamma}^t(\bar{p}, \bar{s}, \bar{\delta})$ for all t. Since, by (A1), public goods are not undesirable, we can now claim that for all t,

$$(6) \qquad (\bar{x}^t_\pi, \bar{x}_g) \in \bar{\gamma}(\bar{p}, \bar{s}, \bar{\delta}) \text{ and } U^t((\bar{x}^t_\pi), \bar{x}_g) \geq U^t(x^t) \text{ for all } x^t \in \bar{\gamma}(\bar{p}, \bar{s}, \bar{\delta}).$$

Once we establish that the allocation $((\bar{x}^t_\pi), \bar{x}_g, (\bar{y}^j))$ is attainable, we can then use the usual argument, as in Debreu (1959, p. 87), to derive from (6), condition (i) of the definition of a Lindahl - Hotelling equilibrium. In fact, condition (iii)' remains the only condition which remains to be verified to complete the proof.

Since $r^t(\bar{p}, (\bar{y}^j)) > 0$ for all t, given the definition of $\bar{p}^t$ and ξ^t we get

$$\bar{p} \cdot (\bar{x}_\pi, \bar{x}_g) \leq \bar{p} \cdot (\bar{y} + \omega),$$

which can be rewritten as

$$(7) \qquad \bar{p} \cdot \big((\bar{x}_\pi, \bar{x}_g) - \sum_h \bar{y}^h - \omega\big) \leq \bar{p} \cdot \bar{y}^c.$$

From the construction of ϕ and the definition of $\bar{y}^c$, we get

$$(8) \qquad (\bar{y}^c + f) = \rho\big((\bar{x}_\pi, \bar{x}_g) - \sum_h \bar{y}^h - \omega + f\big),$$

where ρ is a positive number. This yields $\bar{p} \cdot (\bar{y}^c + f) = \rho\bar{p} \cdot \big((\bar{x}_\pi, \bar{x}_g) - \sum_h \bar{y}^h - \omega + f\big)$. Substituting (7) in this equation we have,

$$(9) \qquad \bar{p} \cdot (\bar{y}^c + f) \leq \rho\bar{p} \cdot (\bar{y}^c + f).$$

Since, we know from the definition of f and k that $\big((\bar{x}_\pi, \bar{x}_g) - \sum_h \bar{y}^h - \omega + f\big) >> 0$, it follows from the construction of ϕ that $\bar{\delta}^c = (\bar{y}^c + f) >> 0$. Now (9) implies that $\rho \geq 1$. Substituting this in (8) and again using the fact that $(\bar{y}^c + f) >> 0$, we get

$$(10) \qquad \bar{y}^c \geq (\bar{x}_\pi, \bar{x}_g) - \sum_h \bar{y}^h - \omega,$$

which is simply condition (iii)'. □

5. Concluding Remarks

We end this paper with some concluding remarks concerning extensions of our result.

There is little doubt that the result presented here can be modified to accommodate non ordered preferences or to provide the existence of a quasi equilibrium under the weaker condition that income is non negative in A5(ii). It is also straightforward to allow for pricing rules more general than the marginal cost pricing rule as, for example, in Bonnisseau - Cornet (1986) and Vohra (1986). An extension to an economy with a continuum of consumers may be technically more demanding but the basic methods of Khan - Vohra (1985) should apply.

A question which has been left open by us, and more generally, by the literature, concerns public inputs i.e. public goods which are also used as inputs by the firms. Hotelling's bridge was, after all, to be used by consumers as well as firms. In this case the cost of public goods have to be appropriately shared by both consumers and firms and our result and its proof would have to be modified to reflect this.

Finally, it is not clear how one would incorporate the impossibility of free disposal (emphasized recently by Cornet (1986)) or the presence of indivisibilities. It is not even clear what the "correct" notion of marginal cost prices would be in this case.

References

Beato, P. and A. Mas-Colell, 1985, On Marginal Cost Pricing with Given Tax-Subsidy Rules, Journal of Economic Theory, 37, 356-365.

Bonnisseau, J. M. and B. Cornet, 1986, Existence of Equilibria when Firms Follow Bounded Losses Pricing Rules, C.O.R.E. Discussion Paper No. 8607.

Bowen, H. R., 1943, The Interpretation of Voting in the Allocation of Economic Resources, Quarterly Journal of Economics, LVIII, 27-48.

Brown, D.J. and G.M. Heal, 1979, Equity, Efficiency and Increasing Returns, Review of Economic Studies, 46, 471-485.

Brown, D.J., G.M. Heal, M. Ali Khan and R. Vohra, 1986, On a General Existence Theorem for Marginal Cost Pricing Equilibria, Journal of Economic Theory, 38, 371-379.

Cornet, B., 1986, The Second Welfare Theorem in Nonconvex Economies, C.O.R.E. Discussion Paper No. 8630.

Debreu, G., 1959, Theory of Value (Wiley, New York).

Foley, D. K., 1970, Lindahl's Solution and the Core of an with Public Goods, Econometrica, 38, 66-72.

Frisch, R., 1939, The Dupuit Taxation Problem, Econometrica, 7, 145-150.

Guesnerie, R., 1975, Pareto Optimality in Non-Convex Economies, Econometrica, 43, 1-29.

Henderson, A. M., 1947, The Pricing of Public Utility Undertakings, The Manchester School of Economic and Social Studies, 15, 223-250.

Hotelling, H., 1938, The General Welfare in Relation to Problems of Taxation and of Railway and Utility Rates, Econometrica, 6, 224-69.

Hotelling, H., 1939, The Relation of Prices to Marginal Costs in an Optimum System, Econometrica, 7, 151-155.

Hurwicz, L. and S. Reiter, 1973, On the Boundedness of the Feasibility Set without Convexity Assumptions, International Economic Review, 14, 580-586.

Kaneko, M., 1977, The Ratio Equilibrium and a Voting Game in a Public Goods Economy, Journal of Economic Theory, 16, 123-136.

Khan, M. Ali and R. Vohra, 1985, On the Existence of Lindahl Equilibria in Economies with a Measure Space of Non-transitive Consumers, Journal of Economic Theory, 36, 319-332.

Khan, M. Ali and R. Vohra, 1987, An Extension of the Second Welfare Theorem to Economies with Non-Convexities and Public Goods, Quarterly Journal of Economics, 102, 223-241.

Lindahl, E., 1919, Just Taxation - A Positive Solution, in: R. Musgrave and A. Peacock eds., Classics in the Theory of Public Finance, 1967, (Martins Press, New York).

Mas-Colell, A. and J. Silvestre, 1985, Cost Share Equilibria: A Lindahlian Approach, University of California, Davis, Working Paper No. 273.

Nash, J., 1951, Non-Cooperative Games, Annals of Mathematics, 54, 286-295.

Samuelson P. A., 1965, The Pure Theory of Public Expenditure, The Review of Economics and Statistics, 36, 387-389.

Vohra, R., 1983, Increasing Returns, Public Goods and General Equilibrium Theory, Ph.D. dissertation, Johns Hopkins University.

Vohra, R., 1985, Optimal Regulation under Fixed Rules for Income Distribution, Brown University Working Paper 85-5, forthcoming, Journal of Economic Theory.

Vohra, R., 1986, On the Existence of Equilibria in Economies with Increasing Returns: A Synthesis, Brown University Paper.